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Orbits Stability and Dimensional Criticality of Cantor Sets

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Abstract—A theorem is presented connecting Golden KAM orbits and the median Hausdorff dimension of a backbone Cantor set at the point of dimensional criticality.

1. STATEMENT OF THE PROBLEM AND GENERAL REMARKS

We seek to first prove and subsequently discuss the mathematical and possible physical implications of the following theorem.

THEOREM. *Let $(\Omega^{(1)})^n$ be a geometrical measure in an n-dimensional space of a multiplicative points set process, and $\Omega^{(1)}$ be given by the Hausdorff dimension of the generating zero set $d_C^{(0)}$, then the average Hausdorff dimension $\langle d \rangle = \left[1/(1-d_C^{(0)})\right]/d_C^{(0)}$ will be exactly equal to the average space dimension $\sim \langle n \rangle = (1+d_C^{(0)})/(1-d_C^{(0)})$, and equivalent to a four-dimensional Cantor set $d_C^{(0)}$ given by $d_n^{(0)} = (1/d_C^{(0)})$, if and only if $d_C^{(0)}$ is equal to the Golden Mean $\phi = \frac{1}{2}\left(\sqrt{5}-1\right)$. That means*

$$\left[\langle d \rangle = \sim \langle n \rangle = \left(\frac{1}{d_C^{(0)}}\right)^3 = d_C^{(4)}\right]\Bigg|_{\phi}.$$

It is clear that the relations expressed in the preceding theorem must be connected to the notion of KAM instability and global chaos in Hamiltonian systems [1]. There, like here, the Golden winding number ratio plays an important role with respect to the considerable stability of certain orbits in the n-dimensional phase space. This is the subject of the following analysis and the subsequent discussion.

2. PROOF OF THE THEOREM

Consider a multiplicative process for which the probability for an event to take place in one dimension is given by a Buffon-like ratio $\Omega^{(1)}$. In an n-dimensional space, we have thus [2]

$$\Omega^{(n)} = \left(\Omega^{(1)}\right)^n. \tag{1}$$

Consequently, the total probability of the additive infinite set [3] formed by $\Omega^{(n)}$ is

$$Z_0 = \sum_{n=0}^{\infty} \left(\Omega^{(1)}\right)^n = \frac{1}{(1-\Omega^{(1)})} \tag{2}$$

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and the normalized probability N is

$$N = \frac{\Omega^{(1)}}{Z_0} = \frac{\Omega^{(1)}}{[1/(1-\Omega^{(1)})]} = \Omega^{(1)}\left(1-\Omega^{(1)}\right). \tag{3}$$

Next, we determine the centre of the area A situated in the Cartesian space $\Gamma(n, Z_n)$ between the trace of the discrete function $Z_n - n$ and its horizontal asymptotic limit [4,5]. This is evidently nothing but the average dimension $\langle n \rangle$. Following the well-known gravity centre theorem, we have thus

$$\tilde{x} = \langle n \rangle \left[\int_0^{-\infty} \left(n^2 \left(\Omega^{(1)}\right)\right) dn\right] \Big/ \left[\int_0^{-\infty} \left(n \left(\Omega^{(1)}\right)^n\right) dn\right]. \tag{4}$$

Evaluating the involved improper integrals, one finds after some manipulations that [4,5]

$$\langle n \rangle = -\frac{2}{\ln \Omega^{(1)}}. \tag{5}$$

Noting that $\Omega^{(1)} < 1$ and expanding $\ln \Omega^{(1)}$, one finds after neglecting higher order terms that [4,5]

$$\sim \langle n \rangle = \frac{\left(1+\Omega^{(1)}\right)}{\left(1-\Omega^{(1)}\right)}. \tag{6}$$

Now it is evident that the reciprocal value of N may be considered an average dimension. In fact, it is an average Hausdorff dimension. This follows immediately from the bijection formula connecting the Menger-Urysohn dimensionality $d_M = n$ with the Hausdorff dimension $d_C^{(n)}$ [3–6]:

$$d_C^{(n)} = \left(\frac{1}{d_C^{(0)}}\right)^{n-1}, \tag{7}$$

where we have assumed that $\Omega^{(1)} \simeq d_C^{(0)}$. In other words,

$$\langle d \rangle = \frac{1}{N} = \frac{1}{[(1-\Omega^{(1)})\,\Omega^{(1)}]}. \tag{8}$$

To find the measure $\Omega^{(1)}$, which satisfies the condition $\sim \langle n \rangle > \langle d \rangle$ we set $\langle d \rangle = \sim \langle n \rangle$. In this way, one finds a quadratic equation

$$\left(\Omega^{(1)}\right)^2 + \Omega^{(1)} - 1 = 0, \tag{9}$$

which has the golden mean as the only possible solution for $\Omega^{(1)} \leq 1$,

$$\Omega^{(1)} = \phi = \frac{1}{2}\left(\sqrt{5}-1\right). \tag{10}$$

To establish the connection to the bijection formula, we set $d_C^{(n)}$ equal to $\langle d \rangle$ and find

$$\left(\frac{1}{d_C^{(0)}}\right)^{n-1} = \frac{1}{[(1-\Omega^{(1)})\,\Omega^{(1)}]}. \tag{11}$$

Evaluating for ϕ one finds that

$$\left(\frac{1}{\phi}\right)^{n-1} = \frac{1}{[(1-\phi)\,\phi]}. \tag{12}$$

Now it is a well-known property of ϕ that $(1/\phi)^3 = 1/[(1-\phi)\,\phi]$ and consequently $d_C^{(n)} \equiv \langle d \rangle$ for $d_C^{(0)} = \Omega^{(1)} = \phi$. We can thus state our final result as follows

$$\left[\langle d \rangle \equiv \sim \langle n \rangle \equiv \left(\frac{1}{d_C^{(0)}}\right)^3 \equiv d_C^{(4)}\right]\Bigg|_{d_C^{(0)}=\phi}, \tag{13}$$

which proves the theorem.

3. TWO FURTHER MEAN VALUES AND TWO UNIVERSAL EXPONENTS

Now to come closer to the principle aim of the present discussion and the relation to KAM [1], we seek to replace $\Omega^{(1)}$ by an average probability $\langle\Omega^{(1)}\rangle$. This is easily performed by applying the gravity centre theorem, this time to $\sim\langle n\rangle$. Proceeding this way, one finds

$$\left\langle\Omega^{(1)}\right\rangle_{(n)}=\left[\int_0^1\frac{\Omega^{(1)}\left(1+\Omega^{(1)}\right)}{1-\Omega^{(1)}}\,d\Omega^{(1)}\right]\bigg/\left[\int_0^1\frac{1+\Omega^{(1)}}{1-\Omega^{(1)}}\,d\Omega^{(1)}\right]. \tag{14}$$

It is possible to avoid the divergence of the integrals involved in $\langle\Omega^{(1)}\rangle_{(n)}$ by remembering that the original expressions for $\langle n\rangle$ was a discrete function. In turn, this entails that a uniform low order biquadratic expression may be regarded as an appropriate and sufficiently accurate estimation. Consequently,

$$\left\langle\Omega^{(1)}_{(n)}\right\rangle=\frac{\int_0^1\left(\Omega^{(1)}+2\left(\Omega^{(1)}\right)^2+2\left(\Omega^{(1)}\right)^3\right)d\Omega^{(1)}}{\int_0^1\left(1+2\Omega^{(1)}+2\left(\Omega^{(1)}\right)^2\right)d\Omega^{(1)}}=\frac{5}{8}=0.625. \tag{15}$$

Now $\langle\Omega^{(1)}\rangle$ is nothing else but the average internal distance $\left\langle d_L^{(0)}\right\rangle$ of a Cantor set. In fact, the result is very close to

$$\left\langle d_C^{(0)}\right\rangle=d_D^{(0)}=0.6296293,$$

found in [7]. Since $d_C^{(0)}=\ln 2/\ln 3=0.630929$, one is even justified in the setting

$$\Omega^{(1)}_{(n)}\simeq d_L^{(0)}\simeq d_D^{(0)}\simeq d_C^{(0)}. \tag{16}$$

It is by now a simple task to drive another gravity centre average dimension, this time using $\langle d\rangle=1/N$. This then leads directly to

$$\left\langle d^{(0)}\right\rangle_{(d)}=\left[\int_0^1\frac{d\Omega^{(1)}}{\left(1+\Omega^{(1)}\right)}\right]\bigg/\left[\int_0^1\frac{d\Omega^{(1)}}{\Omega^{(1)}-\left(\Omega^{(1)}\right)^2}\right]=0.5. \tag{17}$$

Finally, using

$$\sim\langle n\rangle=\frac{1+\Omega^{(1)}}{1-\Omega^{(1)}}=\frac{1+d_C^{(0)}}{1-d_C^{(0)}}. \tag{18}$$

It is easily shown that, $\sim\langle n\rangle=4$ and $\sim\langle n\rangle=5$ one finds

$$d_C^{(0)}\Big|_4=0.6666\quad\text{and}\quad d_C^{(0)}\Big|_5=0.6.$$

These are two well-known exponents of noise fluctuation encountered in a variety of physically quite different problems [8].

4. CONNECTION TO DYNAMIC SYSTEMS

Let us recall first that the quotients $\psi^{(n)}=F_s^{(n)}/F_s^{(n+1)}$ of the Fibonacci series $F_s^{(n+1)}=F^{(n)}+F_s^{(n-1)}$ are given by $\psi^{(1)}=1$, $\psi^{(2)}=0.5$, $\psi^{(3)}=\frac{2}{3}=0.6666$, $\psi^{(4)}=\frac{3}{5}=0.6$, $\psi^{(5)}=\frac{5}{8}=0.625$, $\psi^{(6)}=\frac{8}{13}$, $\psi^{(7)}=\frac{13}{21},\dots$, and that $\psi^{(n)}\to\phi=\frac{1}{2}\left(\sqrt{5}-1\right)$, for $n\to\infty$, where ϕ is the golden mean. The interesting and well-known point here is that $\psi^{(n)}$ are the winding number ratios Φ_n of the phase space orbits of numerous Hamiltonian systems [9]. It is also a well-known fact, confirmed by countless computer calculations that the orbits with $\Phi\equiv\phi$ are the most stable ones. This is of course a result from KAM theory. Now, if we look to the result of the previous paragraph, we can see clearly that all the values found for $d_C^{(0)}$ and $\langle d_C^{(0)} \rangle$ are identical to those of $\psi^{(n)}$. Consequently, we can interpret $\langle d_C^{(0)} \rangle$ also as Φ_n of KAM curves. Viewed in that way, our most important conclusion, namely equation (13), may be reinterpreted in a new light. It is logical that a situation defined by

$$\langle d \rangle < \sim \langle n \rangle, \tag{19}$$

which is only possible for $d_C^{(0)} > \phi$ is a topologically admissible state. By contrast, if

$$\langle d \rangle > \sim \langle n \rangle, \tag{20}$$

which is only possible for $d_C^{(0)} = \phi$, then we are facing a topological inadmissible situation [10]. On the other hand, for $d_C^{(0)} \equiv \phi$, we will have $\langle d \rangle$ filling, "dimensionally speaking" $\sim \langle n \rangle$ completely. This is what we have termed dimensional saturation or ergodic criticality [3–7]. It is self-evident that this situation is far more restrictive than a situation based upon some other measures, such as area or volume filling. This is so because there is no way whatsoever to fit something of higher dimensionality into something with lower dimensionality, even if the "volume" is "equal." It is this topologically pounding property of space that gives an orbit with ϕ as a winding ratio its remarkable stability. Dimensional saturation forces the system, to a certain extent, to ignore perturbations because there is nowhere else to go, so to speak, that is unless n is increased. Finally, we may note that Roessler and his school [11] were among the first pioneers who recognized the importance of high dimensional chaotic dynamics for which the present work may be of relevance.

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